

excursions in Modern Mathematics




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3 The Mathematics of Sharing

3.1 Fair-Division Games

3.2 Two Players: The Divider-Chooser Method

3.3 The Lone-Divider Method

3.4 The Lone-Chooser Method

3.5 The Last-Diminsher Method

3.6 The Method of Sealed Bids

3.7 The Method of Markers

Discrete versus Continuous

As a general rule of thumb, **discrete fair division** is harder to achieve than **continuous fair division** because there is a lot less flexibility in the division process, and discrete fair divisions that are truly fair are only possible under a limited set of conditions. Thus, it is important to keep in mind that while both of the methods we will discuss in the next two sections have limitations, they still are the best methods we have available. Moreover, when they work, both methods work remarkably well and produce surprisingly good fair divisions.

The Method of Sealed Bids

The **method of sealed bids** was originally proposed by Hugo Steinhaus and Bronislaw Knaster around 1948. The best way to illustrate how this method works is by means of an example.

Example 3.9 Settling Grandma's Estate

In her last will and testament, Grandma plays a little joke on her four grandchildren (Art, Betty, Carla, and Dave) by leaving just three items–a cabin in the mountains, a vintage 1955 Rolls Royce, and a Picasso painting–with the stipulation that the items must remain with the grandchildren (not sold to outsiders) and must be divided fairly in equal shares among them. How can we possibly resolve this conundrum? The method of sealed bids will give an ingenious and elegant solution.

Example 3.9 Settling Grandma's Estate

Step 1 (Bidding)

Each of the players makes a bid (in dollars) for each of the items in the estate, giving his or her honest assessment of the actual value of each item. To satisfy the privacy assumption, it is important that the bids are done independently, and no player should be privy to another player's bids before making his or her own. The easiest way to accomplish this is for each player to submit his or her bid in a sealed envelope. When all the bids are in, they are opened.

Example 3.9 Settling Grandma's Estate

Step 1 (Bidding)

Table 3-6 shows each player's bid on each item in the estate.

TABLE 3-6 The Players' Bids

	Art	Betty	Carla	Dave
Cabin	220,000	250,000	211,000	198,000
Rolls Royce	40,000	30,000	47,000	52,000
Picasso	280,000	240,000	234,000	190,000

Example 3.9 Settling Grandma's Estate

Step 2 (Allocation)

Each item will go to the highest bidder for that item. (If there is a tie, the tie can be broken with a coin flip.) In this example the cabin goes to Betty, the vintage Rolls Royce goes to Dave, and the Picasso painting goes to Art. Notice that Carla gets nothing. Not to worry–it all works out at the end! (In this method it is possible for one player to get none of the items and another player to get many or all of the items. Much like in a silent auction, it's a matter of who bids the highest.)

Example 3.9 Settling Grandma's Estate

Step 3 (First Settlement)

It's now time to settle things up. Depending on what items (if any) a player gets in Step 2, he or she will owe money to or be owed money by the estate. To determine how much a player owes or is owed, we first calculate each player's *fair-dollar share* of the estate. A player's *fair-dollar share* is found by adding that player's bids and dividing the total by the number of players.

The last row of Table 3-7 shows the fair-dollar share of each player.

Example 3.9 Settling Grandma's Estate

Step 3 (First Settlement)

TABLE 3-7

	Art	Betty	Carla	Dave
Cabin	220,000	250,000	211,000	198,000
Rolls Royce	40,000	30,000	47,000	52,000
Picasso	280,000	240,000	234,000	190,000
Total	540,000	520,000	492,000	440,000
Fair-dollar share	135,000	130,000	123,000	110,000

Example 3.9 Settling Grandma's Estate

Step 3 (First Settlement)

For example, Art's bids on the three items add up to $540,000. Since there are four equal heirs, Art realizes he is only entitled to one-fourth of that–his fair-dollar share is therefore $135,000.

Example 3.9 Settling Grandma's Estate

Step 3 (First Settlement)

The fair-dollar shares are the baseline for the settlements–if the total value of the items that the player gets in Step 2 is more than his or her fair-dollar share, then the player pays the estate the difference. If the total value of the items that the player gets is less than his or her fair-dollar share, then the player gets the difference in cash.

Here are the details of how the settlement works out for each of our four players.

Example 3.9 Settling Grandma's Estate

Art

As we have seen, Art's fair dollar share is $135,000. At the same time, Art is getting a Picasso painting worth (to him) $280,000, so Art must *pay* the estate the difference of $145,000 ($280,000 – $135,000). The combination of getting the $280,000 Picasso painting but paying $145,000 for it in cash results in Art getting his fair share of the estate.

Example 3.9 Settling Grandma's Estate

Betty

Betty's fair-dollar share is $130,000. Since she is getting the cabin, which she values at $250,000, she must pay the estate the difference of $120,000. By virtue of getting the $250,000 house for $120,000, Betty ends up with her fair share of the estate.

Example 3.9 Settling Grandma's Estate

Carla

Carla's fair-dollar share is $123,000. Since she is getting no items from the estate, she receives her full $123,000 in cash. Clearly, she is getting her fair share of the estate.

Example 3.9 Settling Grandma's Estate

Dave

Dave's fair-dollar share is $110,000. Dave is getting the vintage Rolls, which he values at $52,000, so he has an additional $58,000 coming to him in cash. The Rolls plus the cash constitute Dave's fair share of the estate.

Example 3.9 Settling Grandma's Estate

At this point each of the four heirs has received a fair share, and we might consider our job done, but this is not the end of the story–there is more to come (good news mostly!). If we add Art and Betty's payments to the estate and subtract the payments made by the estate to Carla and Dave, we discover that there is a surplus of $84,000! ($145,000 and $120,000 came in from Art and Betty; $123,000 and $58,000 went out to Carla and Dave.)

Example 3.9 Settling Grandma's Estate

Step 4 (Division of the Surplus)

The surplus is common money that belongs to the estate, and thus to be divided equally among the players. In our example each player's share of the $84,000 surplus is $21,000.

Example 3.9 Settling Grandma's Estate

Step 4 (Division of the Surplus)

The surplus is common money that belongs to the estate, and thus to be divided equally among the players. In our example each player's share of the $84,000 surplus is $21,000.

Step 5 (Final Settlement)

The final settlement is obtained by adding the surplus money to the first settlement obtained in Step 3.

Example 3.9 Settling Grandma's Estate

Step 5 (Final Settlement)

Art:

Gets the Picasso painting and pays the estate $124,000–the original $135,000 he had to pay in Step 3 minus the $21,000 he gets from his share of the surplus.

(Everything done up to this point could be done on paper, but now, finally, real money needs to change hands!)

Example 3.9 Settling Grandma's Estate

Step 5 (Final Settlement)

Betty:

Gets the cabin and has to pay the estate only $99,000 ($120,000 – $21,000).

Carla:

Gets $144,000 in cash ($123,000 + $21,000).

Dave:

Gets the vintage Rolls Royce plus $79,000 ($58,000 + $21,000).

The Method of Sealed Bids

The method of sealed bids works so well because it tweaks a basic idea in economics. In most ordinary transactions there is a buyer and a seller, and the buyer knows the other party is the seller and vice versa. In the method of sealed bids, each player is simultaneously a buyer and a seller, without actually knowing which one until all the bids are opened. This keeps the players honest and, in the long run, works out to everyone's advantage.

The Method of Sealed Bids

At the same time, the method of sealed bids will not work unless the following two important conditions are satisfied.

- Each player must have enough money to play the game. If a player is going to make honest bids on the items, he or she must be prepared to buy some or all of them, which means that he or she may have to pay the estate certain sums of money. If the player does not have this money available, he or she is at a definite disadvantage in playing the game.

The Method of Sealed Bids

- Each player must accept money (if it is a sufficiently large amount) as a substitute for any item. This means that no player can consider any of the items priceless.

The method of sealed bids takes a particularly simple form in the case of two players and one item.

Consider the following example.

Example 3.10 Splitting Up the House

Al and Betty are getting a divorce. The only joint property of any value is their house. Rather than hiring attorneys and going to court to figure out how to split up the house, they agree to give the method of sealed bids a try. Al's bid on the house is $340,000; Betty's bid is $364,000. Their fair-dollar shares of the "estate" are $170,000 and $182,000, respectively. Since Betty is the higher bidder, she gets to keep the house and must pay Al cash for his share.

Example 3.10 Splitting Up the House

The computation of how much cash Betty pays Al can be done in two steps. In the first settlement, Betty owes the estate $182,000. Of this money, $170,000 pays for Al's fair share, leaving a surplus of $12,000 to be split equally between them.

The bottom line is that Betty ends up paying $176,000 to Al for his share of the house, and both come out $6000 ahead.

The Method of Sealed Bids

The method of sealed bids can provide an excellent solution not only to settlements of property in a divorce but also to the equally difficult and often contentious issue of splitting up a partnership. The catch is that in these kinds of splits we can rarely count on the rationality assumption to hold. A divorce or partnership split devoid of emotion, spite, and hard feelings is a rare thing indeed!