

excursions in Modern Mathematics




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seventh edition

3 The Mathematics of Sharing

The Last-Diminisher Method

The last-diminisher method was proposed by Polish mathematicians Stefan Banach and Bronislaw Knaster in the 1940s. The basic idea behind this method is that throughout the game, the set *S* is divided into two pieces–a piece currently "owned" by one of the players (we will call that piece the *C*-piece and the player claiming it the "claimant") and the rest of *S*, "owned" jointly by all the other players.

The Last-Diminisher Method

We will call this latter piece the *R*-piece and the remaining players the “non-claimants.” The tricky part of this method is that the entire arrangement is temporary – as the game progresses, each non-claimant has a chance to become the current claimant (and bump the old claimant back into the non-claimant group) by making changes to the *C*-piece and consequently to the *R*-piece. Thus, the claimant, the non-claimants, the *C*-piece, and the *R*-piece all keep changing throughout the game.

The Last-Diminisher Method

Preliminaries

Before the game starts the players are randomly assigned an order of play (like in a game of Monopoly this can be done by rolling a pair of dice). We will assume that P_1 plays first, P_2 second,…, P_N last, and the players will play in this order throughout the game. The game is played in rounds, and at the end of each round there is one fewer player and a smaller S to be divided.

The Last-Diminisher Method

Round 1

P_1 kicks the game off by "cutting" for herself a 1/*N*th share of *S* (i.e., a share whose value equals 1/*N*th of the value of *S*). This will be the current *C*-piece, and P_1 is its claimant. P_1 does not know whether or not she will end up with this share, so she must be careful that her claim is neither too small (in case she does) nor too large (in case someone else does).

The Last-Diminisher Method

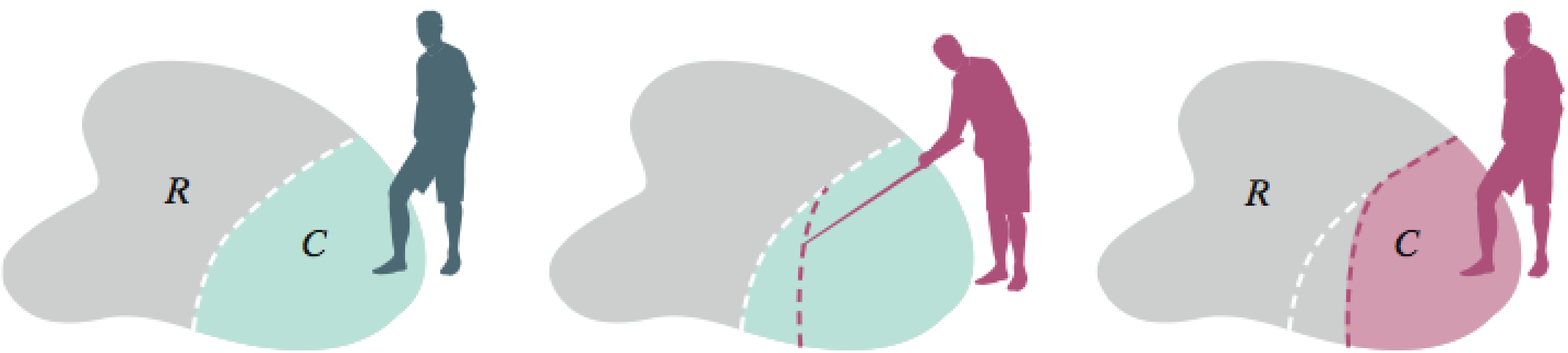
Round 1

P_2 comes next and has a choice: *pass* (remain a nonclaimant) or *diminish* the *C*-piece into a share that is a 1/*N*th share of *S*. Obviously, P_2 can be a diminisher only if he thinks that the value of the current *C*-piece is *more than 1/Nth the value of S*. If P_2 diminishes, several changes take place: P_2 becomes the new claimant; P_1 is bumped back into the nonclaimant group; the diminished *C*-piece becomes the new current *C*-piece; and the “trimmed” piece is added to

The Last-Diminisher Method

Round 1

the old *R*-piece to form a new, larger *R*-piece (there is a lot going on here and it's best to visualize it).



The Last-Diminisher Method

Round 1

P_3 comes next and has exactly the same opportunity as P_2: *pass* or *diminish* the current C-piece. If P_3 passes, then there are no changes and we move on to the next player. If P_3 diminishes (only because in her value system the current C-piece is worth more than $1/N$th of S), she does so by trimming the C-piece to a $1/N$th share of S. The rest of the routine is always the same: The trimmed piece is added to the R-piece, and the previous claimant (P_1 or P_2) is bumped back into the nonclaimant group.

The Last-Diminisher Method

Round 1

The round continues this way, each player in turn having an opportunity to pass or diminish. The last player P_N can also pass or diminish, but if he chooses to diminish, he has a certain advantage over the previous players–knowing that there is no player behind him who could further diminish his claim. In this situation if P_N chooses to be a diminisher, the logical move would be to trim the tiniest possible sliver from the current *C*-piece–a sliver so small that for all practical purposes it has zero value.

The Last-Diminisher Method

Round 1

(Remember that a player's goal is to maximize the size of his or her share.) We will describe this move as "trimming by 0%," although in practice there has to be something trimmed, however small it may be. At the end of Round 1, the current claimant, or last diminisher, gets to keep the *C*-piece (it's her fair share) and is out of the game. The remaining players (the nonclaimants) move on to the next round, to divide the *R*-piece among themselves.

The Last-Diminisher Method

Round 1

At this point everyone is happy–the last diminisher got his or her claimed piece, and the non-claimants are happy to move to the next round, where they will have a chance to divide the *R*-piece among themselves.

The Last-Diminisher Method

Round 2

The *R*-piece becomes the new *S*, and a new version of the game is played with the new *S* and the *N* – 1 remaining players [this means that the new standard for a fair share is a value of 1/(N–1)th or more of the new *S*]. At the end of this round, the last diminisher gets to keep the current *C*-piece and is out of the game.

The Last-Diminisher Method

Rounds 3, 4, and so on

Repeat the process, each time with one fewer player and a smaller *S*, until there are just two players left. At this point, divide the remaining piece between the final two players using the divider-chooser method.

Example 3.8 The Castaways

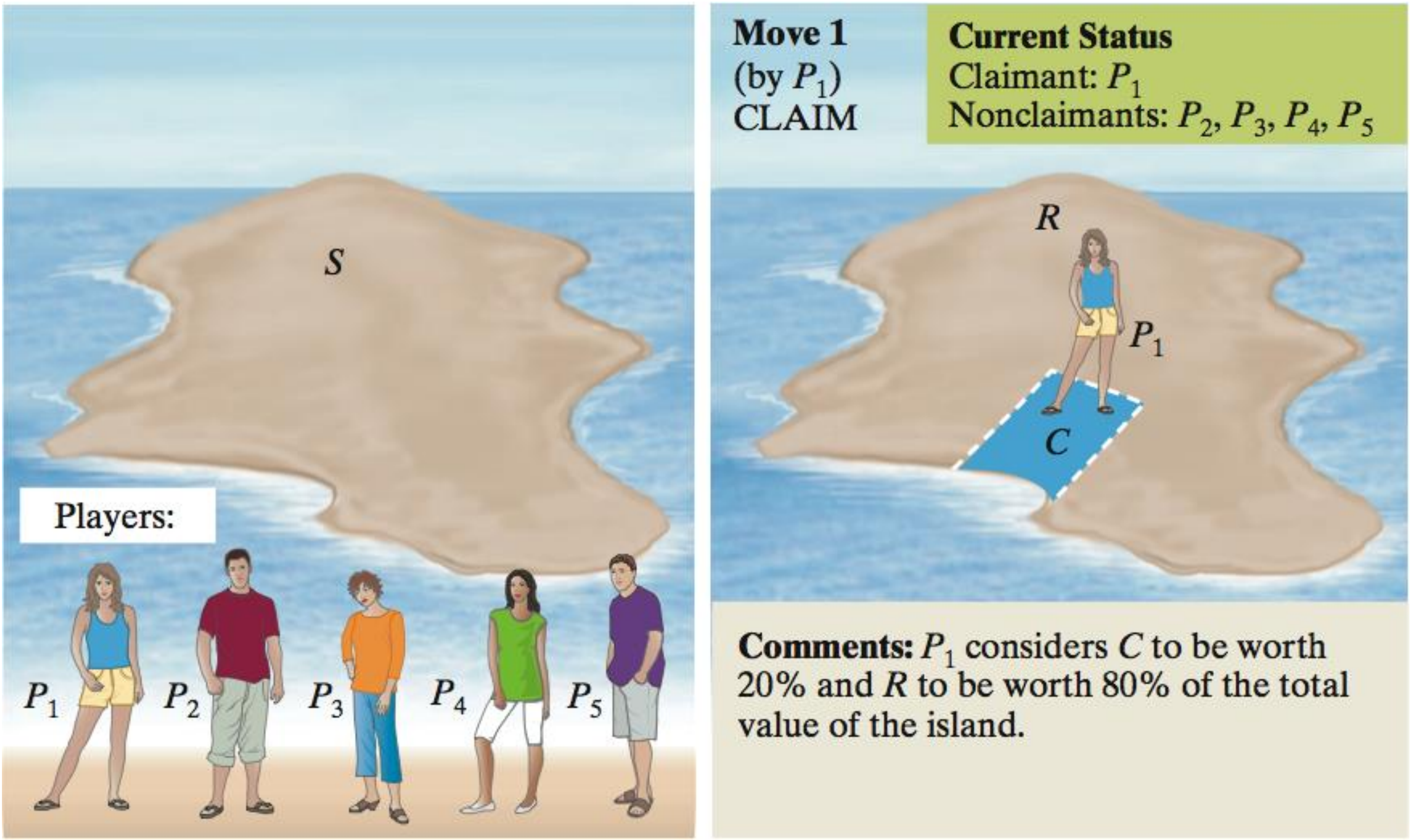
A new reality TV show called *The Castaways* is making its debut this season. In the show five contestants (let's call them P_1, P_2, P_3, P_4, and P_5) are dropped off on a deserted tropical island in the middle of nowhere and left there for a year to manage on their own. After one year, the player who has succeeded and prospered the most wins the million-dollar prize. (The producers are counting on the quarreling, double-crossing, and backbiting among the players to make for great reality TV!)

Example 3.8 The Castaways

In the first episode of the show, the players are instructed to divide up the island among themselves any way they see fit. Instead of quarreling and double-dealing as the producers were hoping for, these five players choose to divide the island using the last-diminisher method. This being reality TV, pictures speak louder than words, and the whole episode unfolds in Figs.3-11 through 3-15.

Example 3.8 The Castaways

Round 1

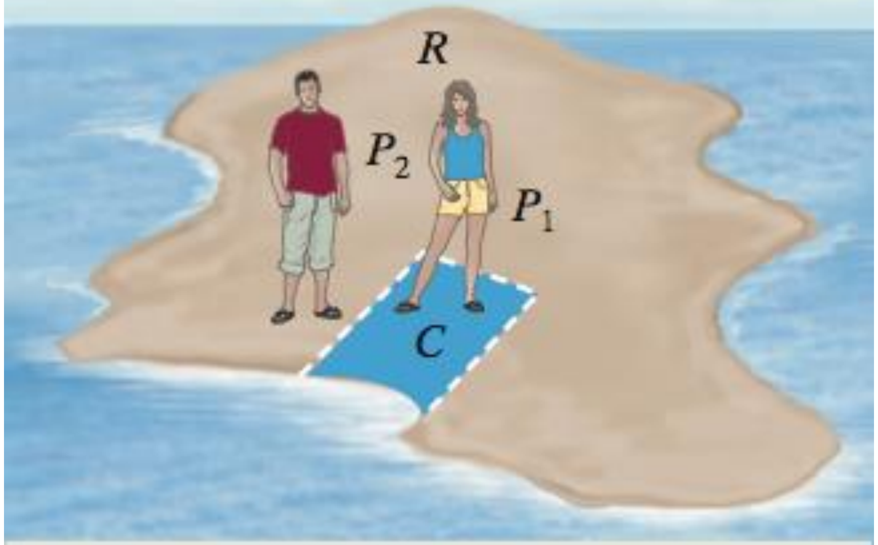


Example 3.8 The Castaways

Round 1

Move 2
(by P_2)
PASS

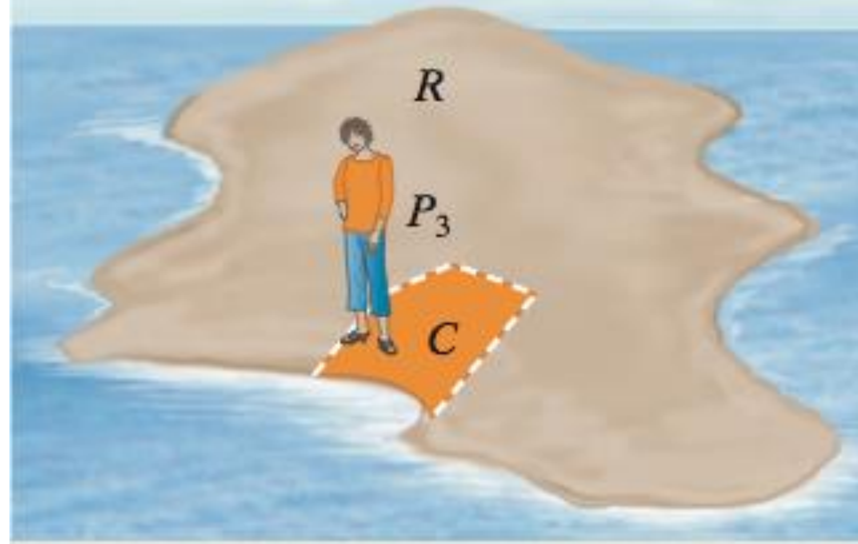
Current Status
Claimant: P_1
Nonclaimants: P_3, P_4, P_5, P_2



Comments: P_2 passes (he considers C to be worth *less* than or equal to 20% of the total value of the island).

Move 3
(by P_3)
DIMINISH

Current Status
Claimant: P_3
Nonclaimants: P_4, P_5, P_2, P_1



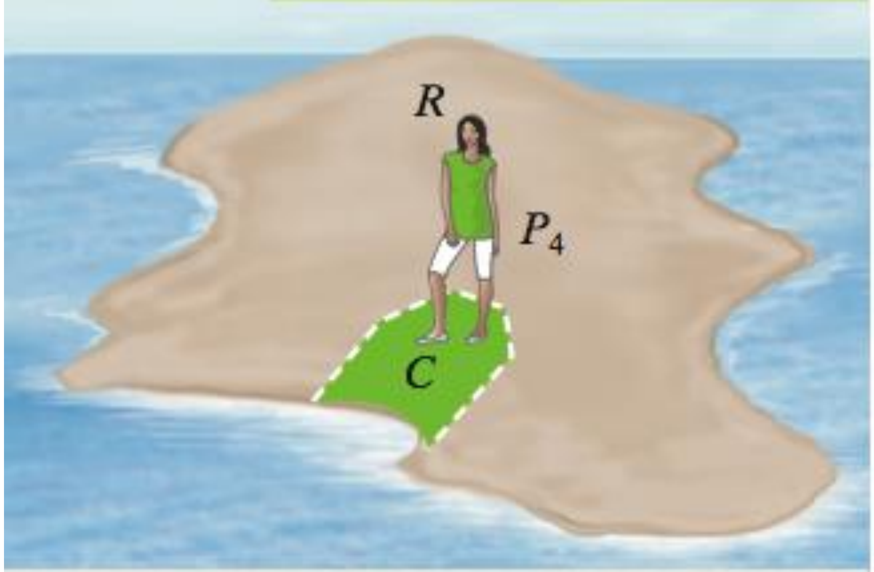
Comments: P_3 considers P_1's claim to be worth *more* than 20% of the total. P_3 diminishes it to a new C worth exactly 20% of the total and becomes its claimant. P_1 becomes a nonclaimant in contention for a fair share of the new R.

Example 3.8 The Castaways

Round 1

Move 4
(by P_4)
DIMINISH

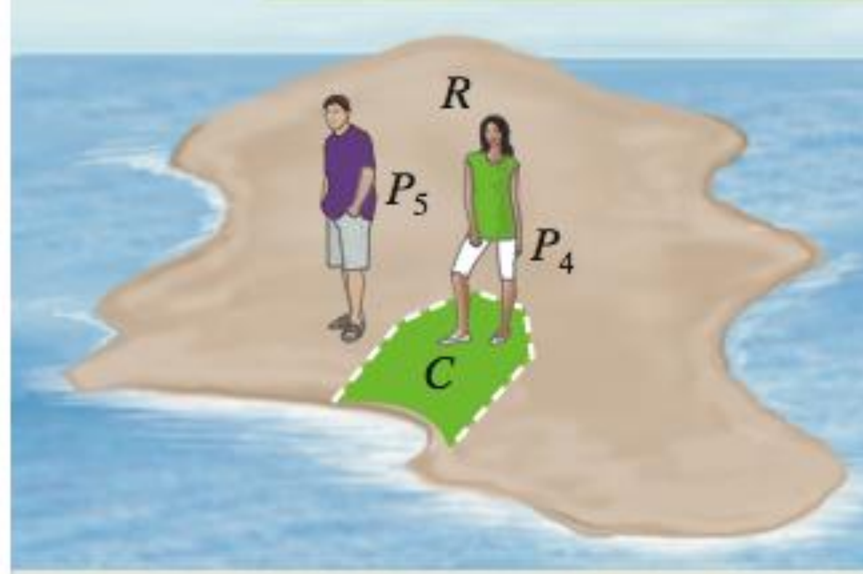
Current Status
Claimant: P_4
Nonclaimants: P_5, P_2, P_1, P_3



Comments: P_4 considers C to be worth *more* than 20% of the total. P_4 diminishes it to a new C worth 20% of the total and becomes its claimant. P_3 becomes a nonclaimant in contention for a fair share of the new R.

Move 5
(by P_5)
PASS

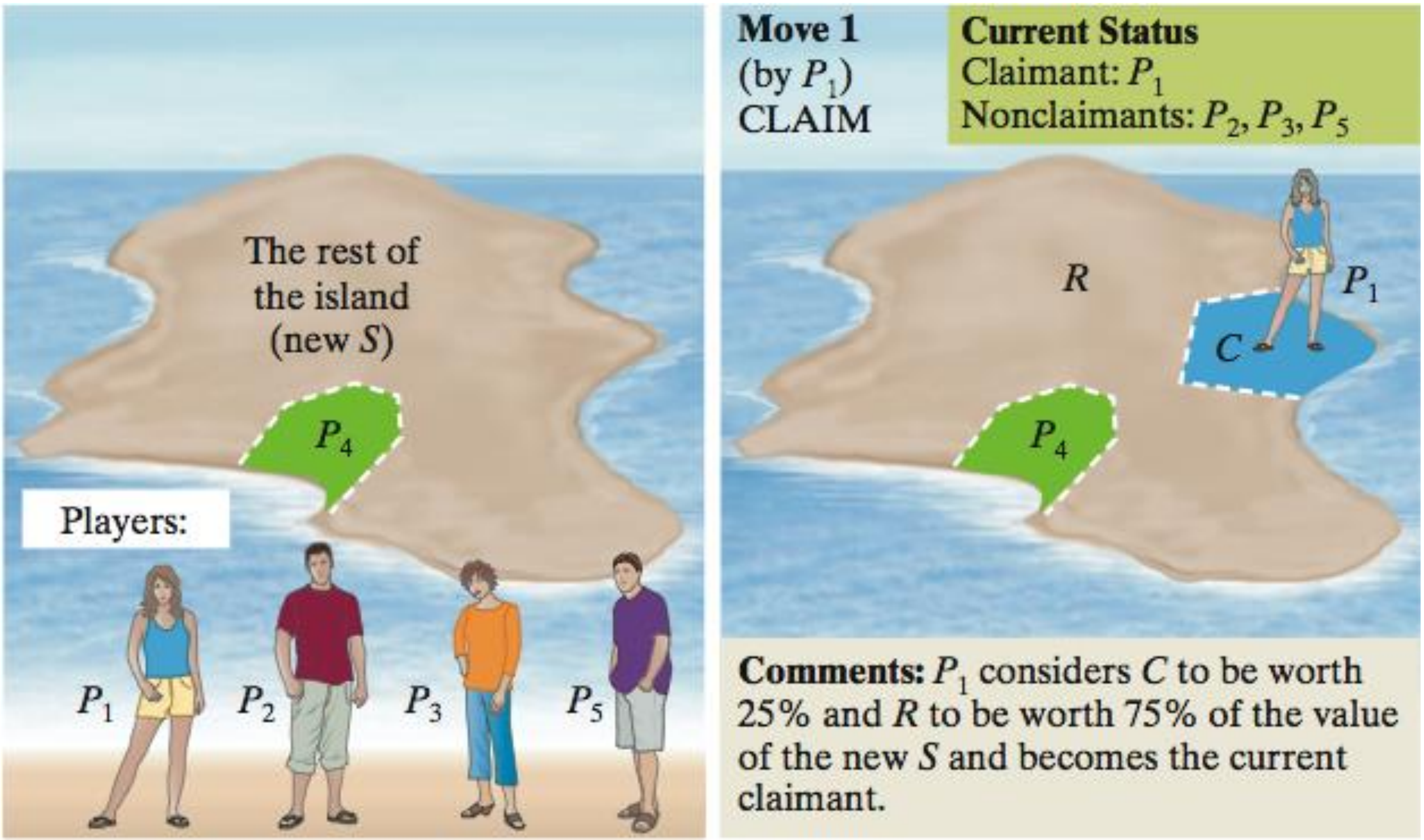
Current Status
Claimant: P_4
Nonclaimants: P_2, P_1, P_3, P_5



Comments: P_5 considers C to be worth *less* than 20% of the total value of the island and passes. All players have now had a chance to diminish or pass. Round 1 is over, with C going to the last diminisher (P_4).

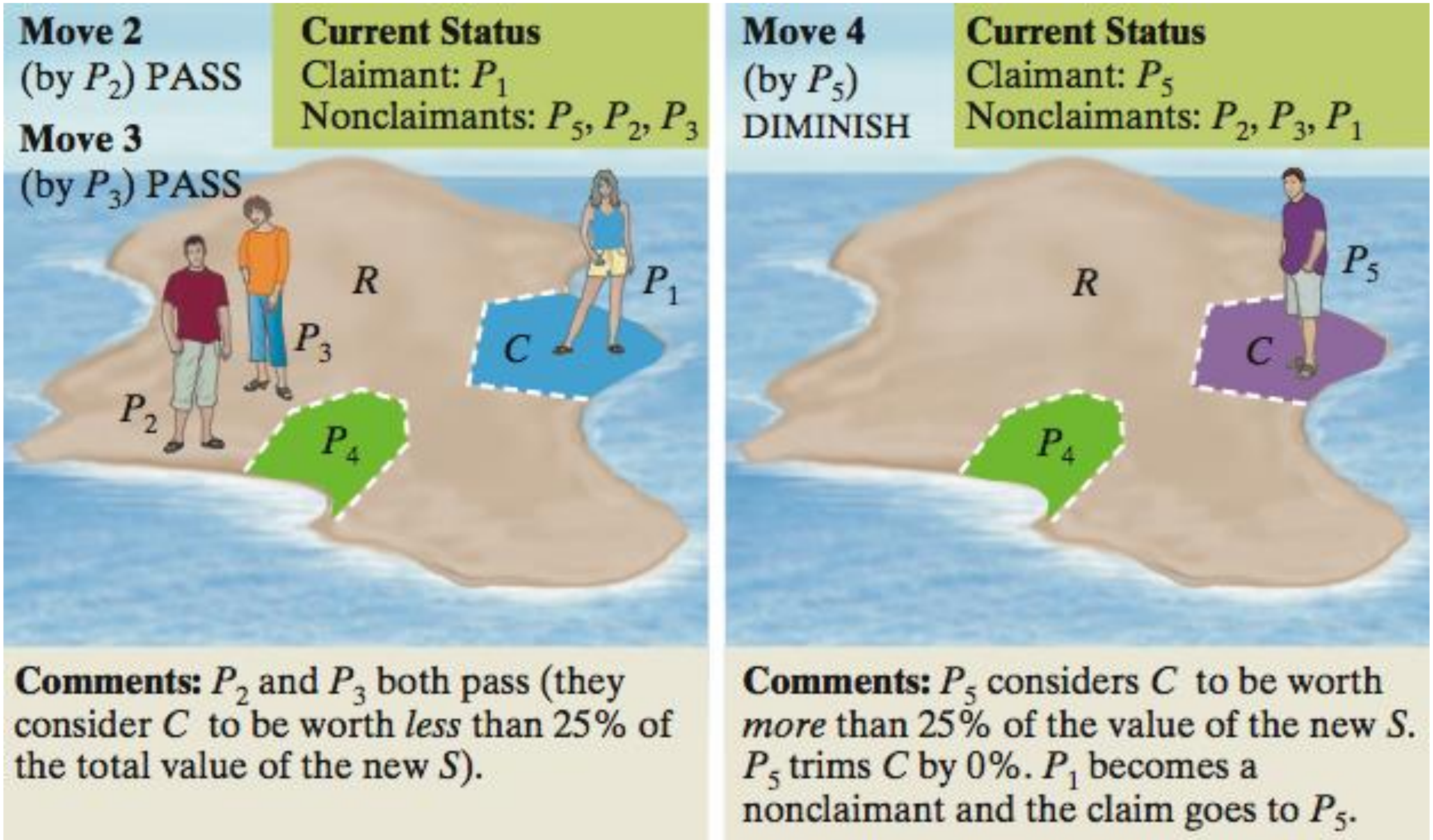
Example 3.8 The Castaways

Round 2



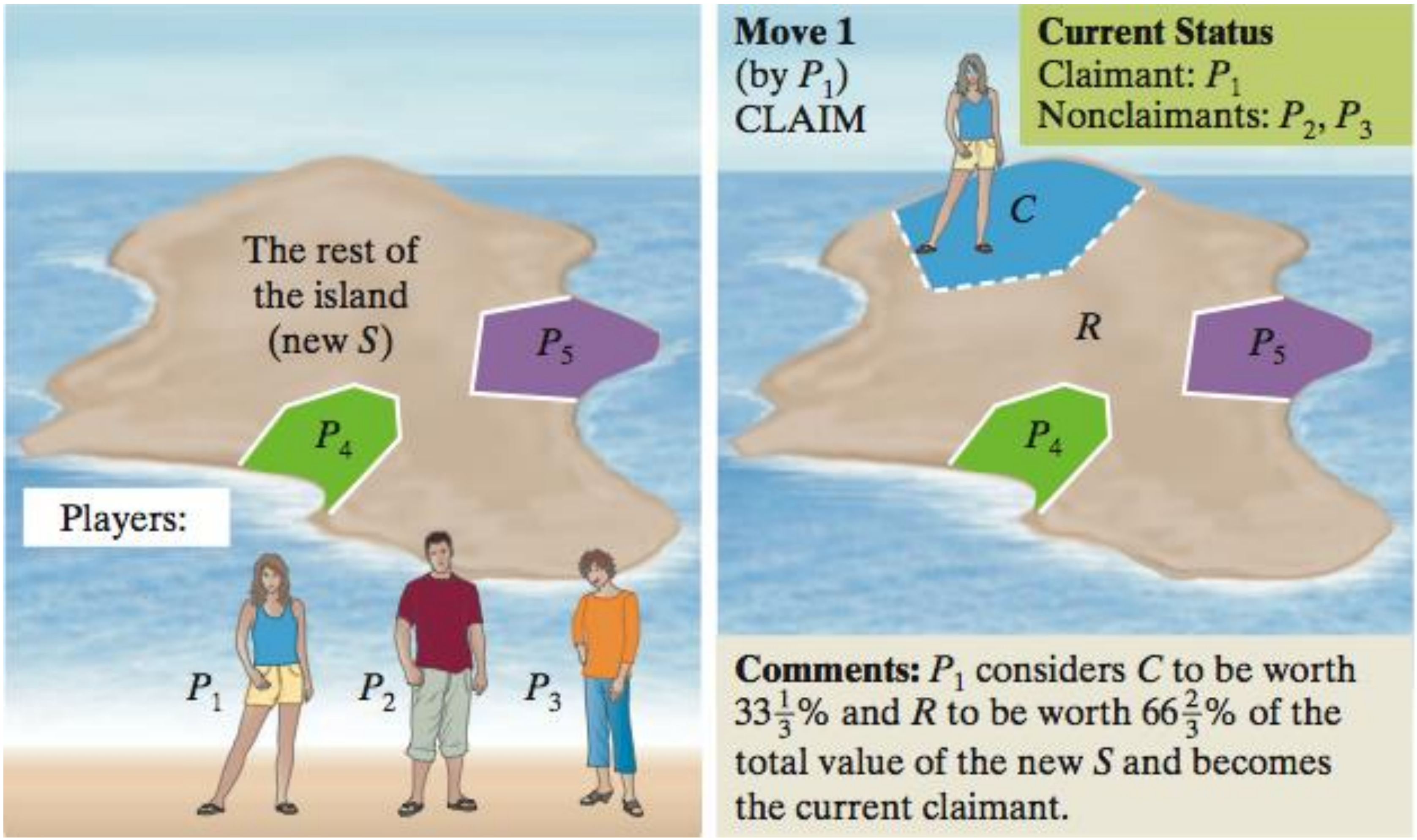
Example 3.8 The Castaways

Round 2



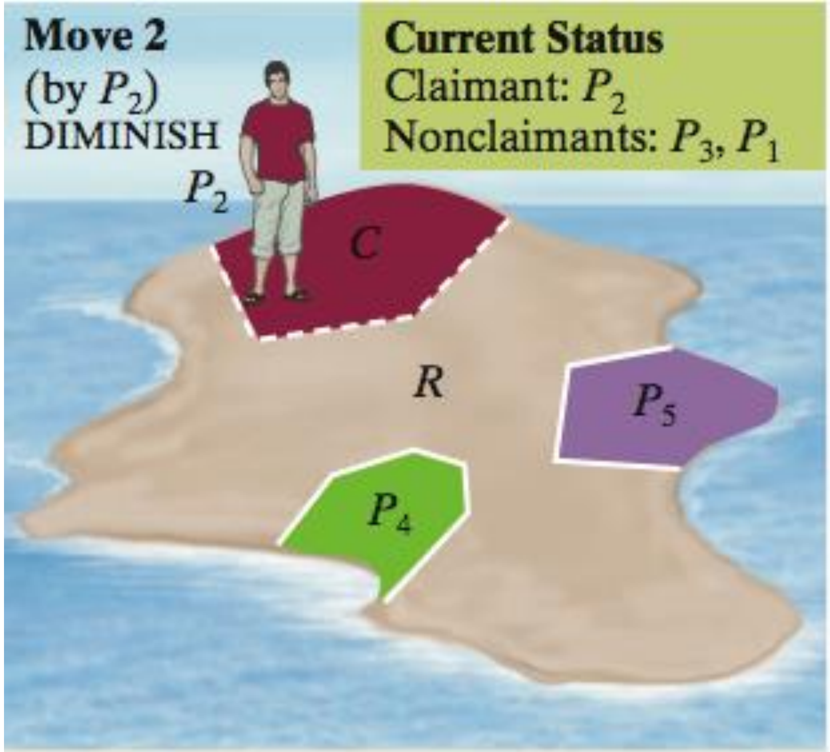
Example 3.8 The Castaways

Round 3

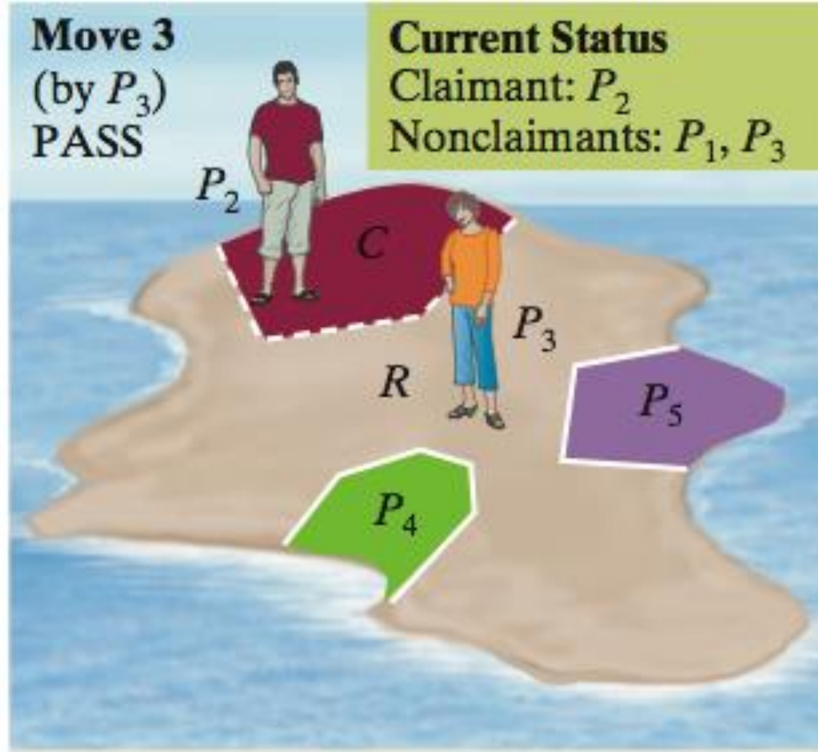


Example 3.8 The Castaways

Round 3



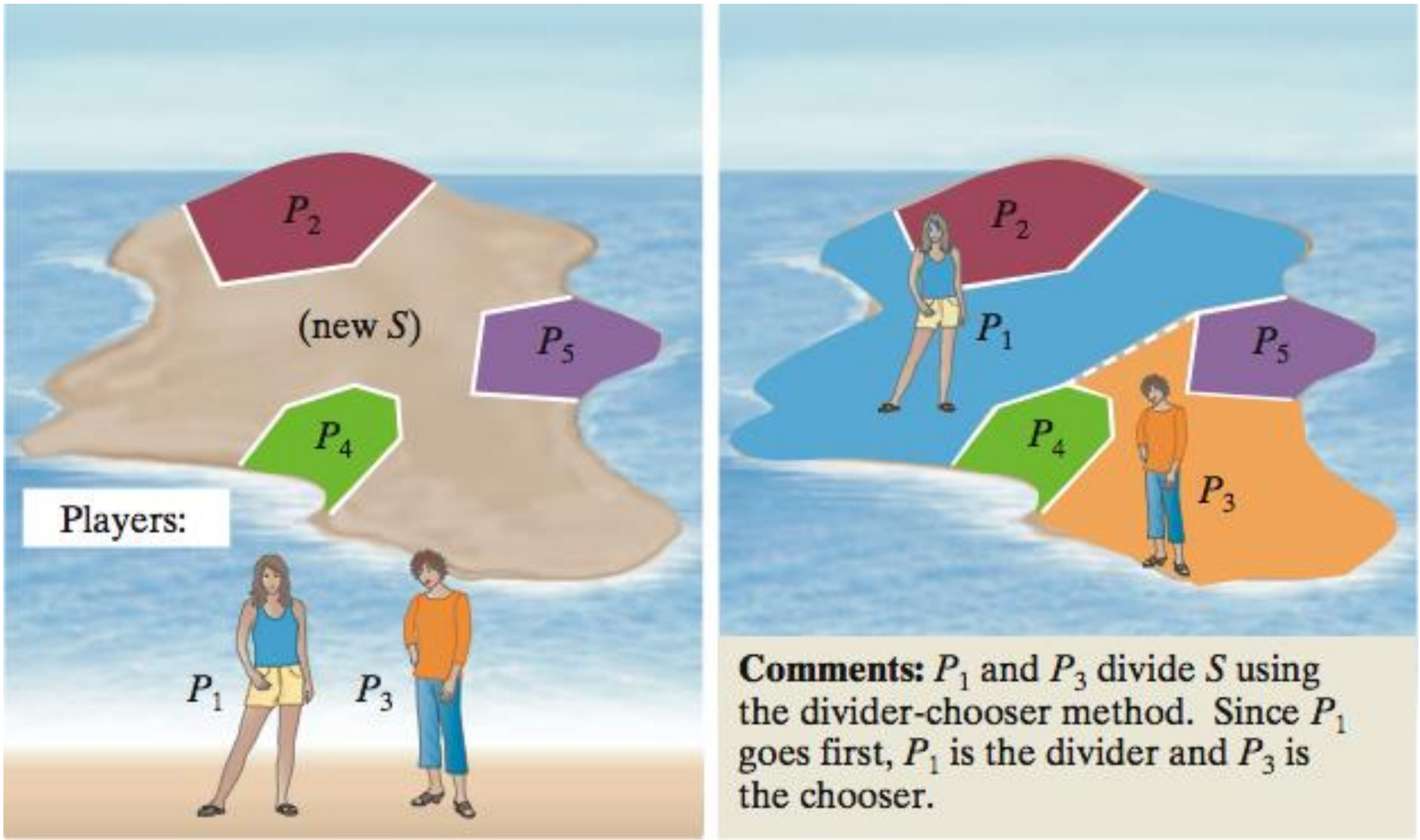
Comments: P_2 considers C to be worth *more* than $33\frac{1}{3}\%$ of the value of S. P_2 diminishes it to a new C worth exactly $33\frac{1}{3}\%$ of the value of S. P_1 goes back to being a nonclaimant.



Comments: P_3 passes (she considers C to be worth *less* than $33\frac{1}{3}\%$ of the value of S). The claim C goes to P_2, the last diminisher.

Example 3.8 The Castaways

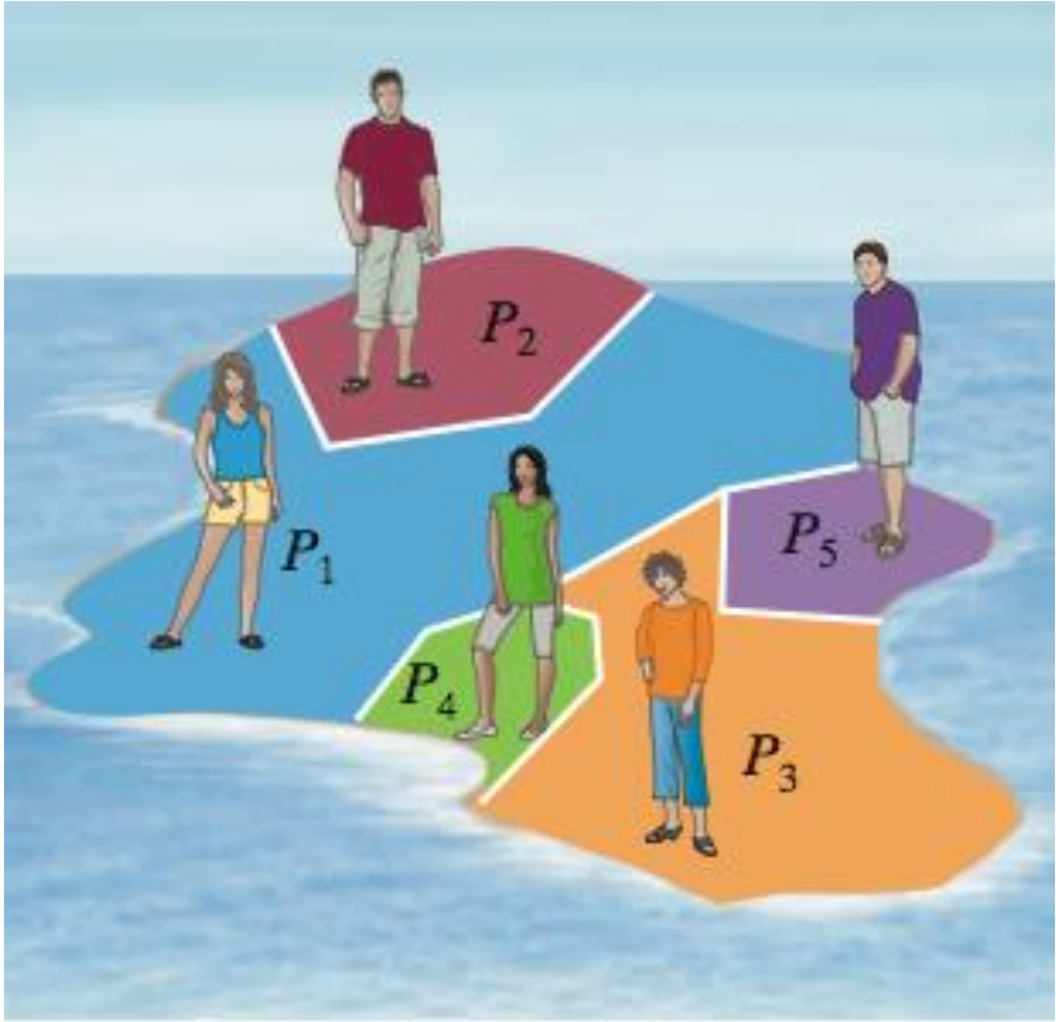
Round 4



Comments: P_1 and P_3 divide S using the divider-chooser method. Since P_1 goes first, P_1 is the divider and P_3 is the chooser.

Example 3.8 The Castaways

The Final Division of the Island



Continuous versus Discrete

In the next two sections we will discuss *discrete* fair-division methods – methods for dividing a booty *S* consisting of indivisible objects such as art, jewels, or candy. As a general rule of thumb, **discrete fair division** is harder to achieve than **continuous fair division** because there is a lot less flexibility in the division process, and discrete fair divisions that are truly fair are only possible under a limited set of conditions. Thus, it is important to keep in mind that while discrete methods have limitations, they still are the best methods we have available.