3 The Mathematics of Sharing

Lone-Divider Method

The first important breakthrough in the mathematics of fair division came in 1943, when Steinhaus came up with a clever way to extend some of the ideas in the divider-chooser method to the case of three players, one of whom plays the role of the divider and the other two who play the role of choosers. Steinhaus' approach was subsequently generalized to any number of players *N* (one divider and *N* – 1 choosers) by Princeton mathematician Harold Kuhn.

Lone-Divider Method for Three Players

Preliminaries

One of the three players will be the divider; the other two players will be choosers. Since it is better to be a chooser than a divider, the decision of who is what is made by a random draw (rolling dice, drawing cards from a deck, etc.). We'll call the divider D and the choosers C_1 and C_2.

Lone-Divider Method for Three Players

Step 1 (Division)

The divider *D* divides the cake into three shares (s_1, s_2, and s_3). *D* will get one of these shares, but at this point does not know which one. (Not knowing which share will be his is critical – it forces *D* to divide the cake into three shares of equal value.)

Lone-Divider Method for Three Players

Step 2 (Bidding)

C_1 declares (usually by writing on a slip of paper) which of the three pieces are fair shares to her. Independently, C_2 does the same. These are the *bids*. A chooser's bid *must list every single piece that he or she considers to be a fair share* (i.e.,worth one-third or more of the cake)–it may be tempting to bid only for the very best piece,but this is a strategy that can easily backfire. To preserve the privacy requirement, it is important that the bids be made independently, without the choosers being privy to each other's bids.

Lone-Divider Method for Three Players

Step 3 (Distribution)

Who gets which piece? The answer, of course, depends on which pieces are listed in the bids. For convenience, we will separate the pieces into two types: *C*-pieces (these are pieces *chosen* by either one or both of the choosers) and *U*-pieces (these are *unwanted* pieces that did not appear in either of the bids).

Lone-Divider Method for Three Players

Step 3 (Distribution) continued

Expressed in terms of value, a *U*-piece is a piece that *both* choosers value at less than 33 1/3% of the cake, and a *C*-piece is a piece that *at least* one of the choosers (maybe both) value at 33 1/3% or more. Depending on the number of *C*-pieces, there are two separate cases to consider.

Lone-Divider Method for Three Players

Case 1

When there are two or more *C*-pieces, there is always a way to give each chooser a different piece from among the pieces listed in her bid. (The details will be covered in Examples 3.2 and 3.3.) Once each chooser gets her piece, the divider gets the last remaining piece. At this point every player has received a fair share, and a fair division has been accomplished.

Lone-Divider Method for Three Players

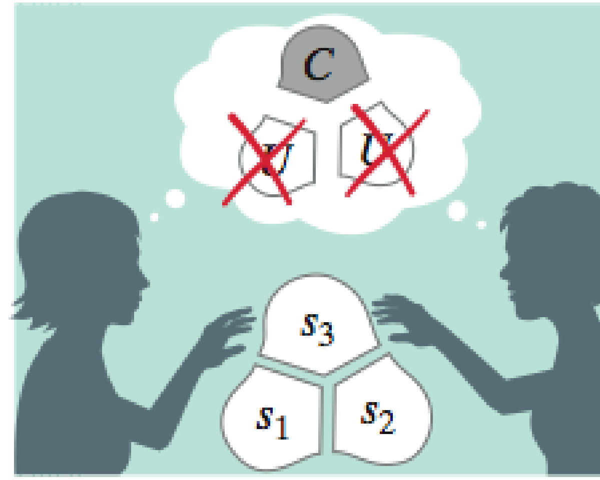
Case 1

(Sometimes we might end up in a situation in which C_1 likes C_2's piece better than her own and vice versa. In that case it is perfectly reasonable to add a final, informal step and let them swap pieces–this would make each of them happier than they already were, and who could be against that?)

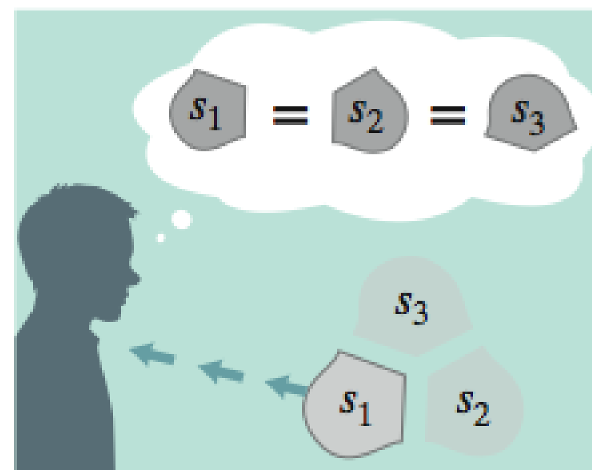
Lone-Divider Method for Three Players

Case 2

When there is only one *C*-piece, we have a bit of a problem because it means that both choosers are bidding for the very same piece.



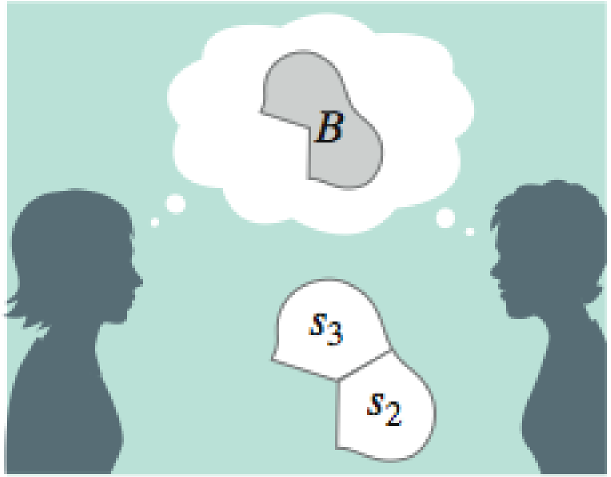
The solution requires a little more creativity. First, we take care of the divider *D*–to whom all pieces are equal in value–by giving him one of the pieces that neither chooser wants.



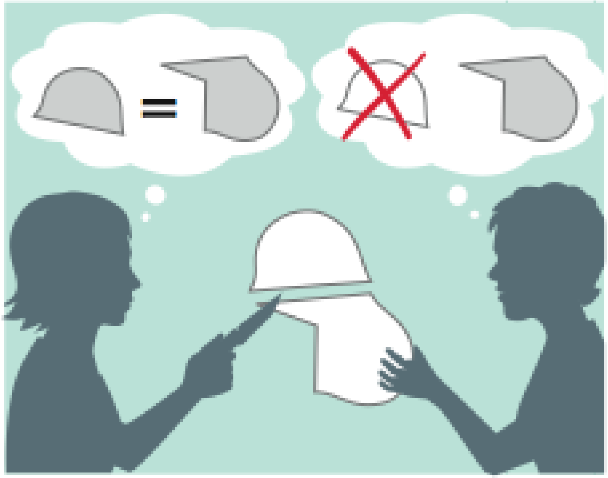
Lone-Divider Method for Three Players

Case 2

After *D* gets his piece,the two pieces left (the *C*-piece and the remaining *U*-piece) are recombined into one piece that we call the *B*-piece.



We can revert to the divider-chooser method to finish the fair division: one player cuts the *B*-piece into two pieces; the other player chooses the piece she likes better.



Lone-Divider Method for Three Players

Case 2

This process results in a fair division of the cake because it guarantees fair shares for all players. We know that *D* ends up with a fair share by the very fact that *D* did the original division, but what about C_1 and C_2? The key observation is that in the eyes of both C_1 and C_2 the *B*-piece is worth *more than two-thirds of the value of the original cake* (think of the *B*-piece as 100% of the original cake minus a *U*-piece worth less than 33 1/3%), so when we divide it fairly into *two* shares, each party is guaranteed *more than one-third* of the original cake.

Example 3.2 Lone Divider with 3 Players (Case 1, Version 1)

Dale, Cindy, and Cher are dividing a cake using Steinhaus's lone-divider method. They draw cards from a well-shuffled deck of cards, and Dale draws the low card (bad luck!) and has to be the divider.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-1 shows the values of the three pieces in the eyes of each of the players.

Example 3.2 Lone Divider with 3 Players (Case 1, Version 1)

TABLE 3-1

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	35%	10%	55%
Cher	40%	25%	35%

Step 2 (Bidding)

We can assume that Cindy's bid list is $\{s_1, s_3\}$ and Cher's bid list is also $\{s_1, s_3\}$.

Example 3.2 Lone Divider with 3 Players (Case 1, Version 1)

Step 3 (Distribution)

The *C*-pieces are s_1 and s_3. There are two possible distributions. One distribution would be: Cindy gets s_1, Cher gets s_3, and Dale gets s_2. An even better distribution (the *optimal* distribution) would be: Cindy gets s_3, Cher gets s_1, and Dale gets s_2. In the case of the first distribution, both Cindy and Cher would benefit by swapping pieces, and there is no rational reason why they would not do so.

Example 3.2 Lone Divider with 3 Players (Case 1, Version 1)

Step 3 **(Distribution)** continued

Thus, using the rationality assumption, we can conclude that in either case the final result will be the same: Cindy gets s_3, Cher gets s_1, and Dale gets s_2.

Example 3.3 Lone Divider with 3 Players (Case 1, Version 2)

We'll use the same setup as in Example 3.2–Dale is the divider, Cindy and Cher are the choosers.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-2 shows the values of the three pieces in the eyes of each of the players.

Example 3.3 Lone Divider with 3 Players (Case 1, Version 2)

TABLE 3-2

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	30%	40%	30%
Cher	60%	15%	25%

Step 2 (Bidding)

Here Cindy's bid list is $\{s_2\}$ only, and Cher's bid list is $\{s_1\}$ only.

Example 3.3 Lone Divider with 3 Players (Case 1, Version 2)

Step 3 (Distribution)

This is the simplest of all situations, as there is only one possible distribution of the pieces: Cindy gets s_2, Cher gets s_1, and Dale gets s_3.

Example 3.4 Lone Divider with 3 Players (Case 2)

The gang of Examples 3.2 and 3.3 are back at it again.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-3 shows the values of the three pieces in the eyes of each of the players.

Example 3.4 Lone Divider with 3 Players (Case 2)

TABLE 3-3

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	20%	30%	50%
Cher	10%	20%	70%

Step 2 (Bidding)

Here Cindy's and Cher's bid list consists of just $\{s_3\}$.

Example 3.4 Lone Divider with 3 Players (Case 2)

Step 3 (Distribution)

The only C-piece is s_3. Cindy and Cher talk it over, and without giving away any other information agree that of the two *U*-pieces, s_1 is the least desirable, so they all agree that Dale gets s_1. (Dale doesn't care which of the three pieces he gets, so he has no rational objection.) The remaining pieces (s_2 and s_3) are then recombined to form the B-piece, to be divided between Cindy and Cher using the divider-chooser method (one of them divides

Example 3.2 Lone Divider with 3 Players (Case 1, Version 1)

Step 3 (Distribution) continued

the *B*-piece into two shares, the other one chooses the share she likes better). Regardless of how this plays out, both of them will get a very healthy share of the cake: Cindy will end up with a piece worth at least 40% of the original cake (the *B*-piece is worth 80% of the original cake to Cindy), and Cher will end up with a piece worth at least 45% of the original cake (the *B*-piece is worth 90% of the original cake to Cher).

The Lone-Divider Method for More Than Three Players

The first two steps of Kuhn's method are a straightforward generalization of Steinhaus's lone-divider method for three players, but the distribution step requires some fairly sophisticated mathematical ideas and is rather difficult to describe in full generality, so we will only give an outline here and will illustrate the details with a couple of examples for $N = 4$ players.

The Lone-Divider Method for More Than Three Players

Preliminaries

One of the players is chosen to be the divider D, and the remaining N – 1 players are all going to be choosers. As always, it's better to be a chooser than a divider, so the decision should be made by a random draw.

The Lone-Divider Method for More Than Three Players

Step 1 (Division)

The divider *D* divides the set *S* into *N* shares $s_1, s_2, s_3, ..., s_N$. *D* is guaranteed of getting one of these shares, but doesn't know which one.

The Lone-Divider Method for More Than Three Players

Step 2 (Bidding)

Each of the $N - 1$ choosers independently submits a bid list consisting of every share that he or she considers to be a fair share (i.e., worth $1/N$th or more of the booty S).

The Lone-Divider Method for More Than Three Players

Step 3 (Distribution)

The bid lists are opened. Much as we did with three players, we will have to consider two separate cases, depending on how these bid lists turn out.

The Lone-Divider Method for More Than Three Players

Case 1.

If there is a way to assign a different share to each of the N – 1 choosers, then that should be done. (Needless to say, the share assigned to a chooser should be from his or her bid list.) The divider, to whom all shares are presumed to be of equal value, gets the last unassigned share. At the end, players may choose to swap pieces if they want.

The Lone-Divider Method for More Than Three Players

Case 2.

There is a *standoff*–in other words, there are two choosers both bidding for just one share, or three choosers bidding for just two shares, or K choosers bidding for less than K shares. This is a much more complicated case, and what follows is a rough sketch of what to do. To resolve a standoff, we first set aside the shares involved in the standoff from the remaining shares.

The Lone-Divider Method for More Than Three Players

Case 2.

Likewise, the players involved in the standoff are temporarily separated from the rest. Each of the remaining players (including the divider) can be assigned a fair share from among the remaining shares and sent packing. All the shares left are recombined into a new booty *S* to be divided among the players involved in the standoff, and the process starts all over again.

Example 3.5 Lone Divider with 4 Players (Case 1)

We have one divider, Demi, and three choosers, Chan, Chloe, and Chris.

Step 1 (Division)

Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-4 shows how each of the players values each of the four shares. Remember that the information on each row of Table 3-4 is private and known only to that player.

Example 3.5 Lone Divider with 4 Players (Case 1)

TABLE 3-4

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	30%	20%	35%	15%
Chloe	20%	20%	40%	20%
Chris	25%	20%	20%	35%

Step 2 (Bidding)

Chan's bid list is $\{s_1, s_3\}$; Chloe's bid list is $\{s_3\}$ only; Chris's bid list is $\{s_1, s_4\}$.

Example 3.5 Lone Divider with 4 Players (Case 1)

Step 3 (Distribution)

The bid lists are opened. It is clear that for starters Chloe must get s_3 – there is no other option. This forces the rest of the distribution: s_1 must then go to Chan, and s_4 goes to Chris. Finally, we give the last remaining piece, s_2, to Demi.

Example 3.5 Lone Divider with 4 Players (Case 1)

This distribution results in a fair division of the cake, although it is not entirely “envy-free” Chan wishes he had Chloe’s piece (35% is better than 30%) but Chloe is not about to trade pieces with him, so he is stuck with s_1. (From a strictly rational point of view, Chan has no reason to gripe–he did not get the best piece, but got a piece worth 30% of the total, better than the 25% he is entitled to.)

Example 3.6 Lone Divider with 4 Players (Case 2)

Once again, we will let Demi be the divider and Chan, Chloe, and Chris be the three choosers (same players, different game).

Step 1 (Division)

Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-5 shows how each of the players values each of the four shares.

Example 3.6 Lone Divider with 4 Players (Case 2)

TABLE 3-5

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	20%	20%	20%	40%
Chloe	15%	35%	30%	20%
Chris	22%	23%	20%	35%

Step 2 (Bidding)

Chan's bid list is $\{s_4\}$; Chloe's bid list is $\{s_2, s_3\}$ only; Chris's bid list is $\{s_4\}$.

Example 3.6 Lone Divider with 4 Players (Case 2)

Step 3 (Distribution)

The bid lists are opened, and the players can see that there is a standoff brewing on the horizon–Chan and Chris are both bidding for s_4. The first step is to set aside and assign Chloe and Demi a fair share from s_1, s_2, and s_4. Chloe could be given either s_2 or s_3. (She would rather have s_2,of course, but it's not for her to decide.)

Example 3.6 Lone Divider with 4 Players (Case 2)

Step 3 (Distribution)

A coin toss is used to determine which one. Let's say Chloe ends up with s_3 (bad luck!). Demi could be now given either s_1 or s_2. Another coin toss, and Demi ends up with s_1. The final move is ... you guessed it!– recombine s_2 and s_4 into a single piece to be divided between Chan and Chris using the divider-chooser method.

Example 3.6 Lone Divider with 4 Players (Case 2)

Step 3 (Distribution)

Since (s_2 + s_4) is worth 60% to Chan and 58% to Chris (you can check it out in Table 3-5), regardless of how this final division plays out they are both guaranteed a final share worth more than 25% of the cake.

Mission accomplished! We have produced a fair division of the cake.